GENETIC ALGORITHM AND METHOD MONTE CARLO APPROACH FOR PROSPECT THEORY MODEL WITH CARDINALITY CONSTRAINT

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The reality of modern finance is that a major paradigm shift is underway. This is a common issue for all spheres of knowledge in the modern world because of international integration and sophistication of new technologies. The more science develops the more new questions appear. Chances are that "the new financial paradigm" will combine neoclassical and behavioural elements [1].

Behavioural finance is the study of how psychology impacts financial decisions in households, markets and organisations. Behavioural finance does not assume rational agents or frictionless markets. It suggests that the institutional environment is vitally important. The starting point is bounded rationality [1].

The recent financial crisis has shown the shortcomings of individual market instruments and the low level of validity investment decisions. This largely can be explained by dismissive investors' attitude to assess the real risks that are perceived by them in the intuitive level.

One of the most prominent alternatives to the mean-variance theory and expected utility theory is prospect theory. This theory is one of the economic theories which incorporates real human decision patterns and psychology into choice behavior. It was created in order to estimate the risk involving gains and losses. Within this framework, the individuals estimate the losses and gains utility function subjectively. For this purpose they use a starting point (or reference point or "status quo") so, that:

- if a portfolio return value is equal to this number, then it means that the investor obtains a zero gain,
- if a portfolio return value is higher, then the investor considers that he gained from the portfolio investment,
- if it is lower, then he lost.

Prospect theory is important for decision making under uncertainty. It departs from the traditional expected utility framework in important ways. It provides psychological underpinnings for the behavioural approaches to portfolio selection that are quite different from the traditional approaches such as the mean variance framework. Prospect theory was developed by two psychologists, Daniel Kahneman and Amos Tversky, and published in the Econometrica in 1979 [2].

Traditional finance theory assumes that investors make a decision under uncertainty by maximizing expected utility of wealth or consumption. The expected utility theory is mathematically elegant and is a rational-based framework built upon axioms. However, the underlying assumptions have been shown by many studies to be an inaccurate description of how people actually behave when choosing among risky alternatives.

Kahneman and Tversky experimentally obtained the utility (value) function which was dependent on the initial value deviation. This value function is convex for the gains and non-convex for the losses [2]. This means risk aversion is associated with the cases of the gains and risk inclination with the losses. It is worth mentioning that the aforementioned function has the steepest gradient for losses.

The original prospect theory choice process and objective function consists of two phases and corresponding functions. The choice process under prospect theory starts with the editing phase, followed by the evaluation of edited prospects, and finally the alternative with the highest value is chosen [3].

During the editing phase, agents discriminate gains and losses. The agents attach a subjective value to the gamble relative to a reference point r_0. They assume the value function:

$$v(r) = \begin{cases} (r - r_0)^{\alpha}, \ if \ r \geq r_0 \\ \lambda(r_0 - r)^{\beta}, \ if \ r < r_0 \end{cases} \quad (1)$$

where $\alpha, \beta, \lambda > 0$. Kahneman and Tversky found in their experiments that $\alpha = \beta = 0.88$ and $\lambda = 2.25$ [4]. These coefficients characterise the level of risk aversion (α and β) and the level of loss aversion (λ).

Standard utility functions have been replaced by the prospect theory value function. It has two parts. The part in the gain domain is concave and the part in the loss domain is convex, capturing the risk-averse tendency for gains and risk-seeking tendency for losses by many decision makers [5].

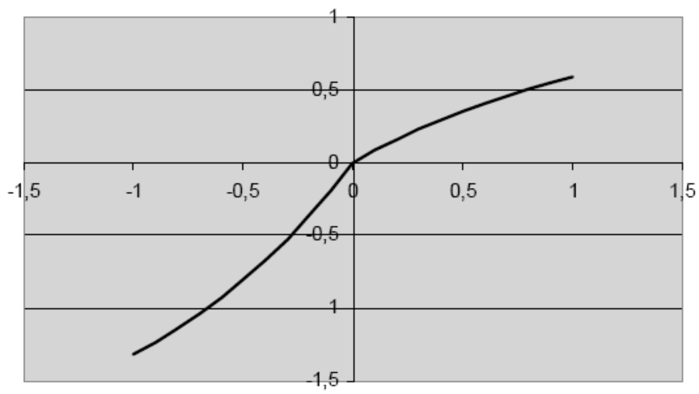

Fig. 1. Prospect Theory Value Function with $\alpha = \beta = 0.88$ and $\lambda = 2.25$

The next stage is the evaluating phase where an investor calculates the prospect theory utility based on the potential outcomes and their respective probabilities, and then chooses the alternative having a higher utility as follows:

$$PT_U = \sum_{i=1}^{n} \pi(x) v(x_i) \quad (2)$$

Investors perform additional mental adjustments in the original probability function $p = f(x)$, defining the prospect theory probability weighting function $\pi(p)$. According to this consider the probability weighting function:

$$\pi(x) = \frac{p^{\gamma}}{\left(p^{\gamma} + (1-p)^{\gamma}\right)^{\frac{1}{\gamma}}} \quad (3)$$

where γ is the adjustment factor.

The probability weighting function, which is based on the observation that most people tend to overweigh small probabilities and underweigh large probabilities. Although the original formulation of prospect theory proposed by Kahneman and Tversky (1979) was only defined mostly for lotteries with two non-zero outcomes, it can be generalised to n outcomes. Generalisations have been used by various authors (see, for example [6], [7], [8]).

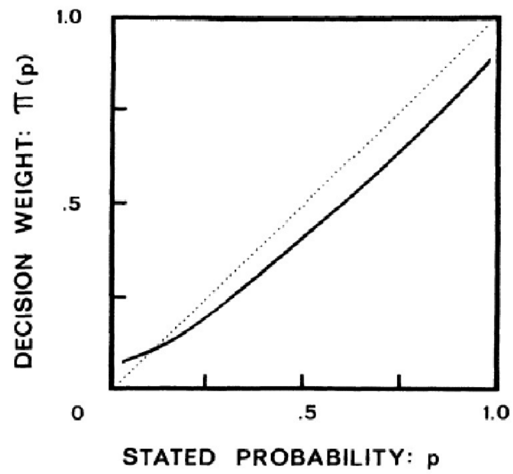

Fig. 2. A Hypothetical Prospect Theory Weight Function
by Kahneman and Tversky 1979

This formulation illustrates all four elements of prospect theory:

– reference dependence,
– loss aversion,
– diminishing sensitivity, and
– probability weighting.

The portfolio optimisation problem is a question of how to determine an amount (proportion, weight) of money to invest in each type of asset within the portfolio in order to receive the highest possible return (or utility) subject to appropriate level of risk by the end of the investment period. At least one constraint is known that the sum of the weights of the securities must be equal to one.

In this paper we consider prospect theory model with cardinality constraint in the form of portfolio optimisation problem. It is known that adding limit for number of assets in portfolio change the classical prospect theory model into mixed integer non-linear programming problem which is NP-complete [9]. To solve this problem we use heuristics namely genetic algorithm and method Monte Carlo dealing with prospect theory model, basing on the fact that this model is hard to solve and standard numerical approaches is incapable to get a quality solution. The complexity of the problem comes from the non linearity of the objective function as well as non linear constraints of the model.

Let:

N – number of assets

S – number of scenarios (time periods)

ρ_j – probability of scenario $\sum \rho_s = 1$

$\bar{r}_i$ – expected return of asset i

r_{is} – return of asset i in scenario $s, (1 \le i \le N)(1 \le s \le S)$

$\omega_i \ge 0$ – weight of asset i in portfolio

$x = (\omega_1, \dots, \omega_N)$ – portfolio and $\sum_{i=1}^{N} \omega_i = 1$

$X = \{x = (\omega_1, \dots, \omega_N) \in \mathrm{R}^N : \forall i\}$ – set of all portfolios

$r_s(x)$ – return of portfolio x in scenario j with respect probability ρ_s

d – desirable level of return.

Using the notation given above we formulate the prospect theory model with cardinality constraint:

$$maximise(C)PT_{cc} = \sum_{j=1}^{n} \pi_j v(r_j(x)), \quad (4)$$

Subject to:

$$r(x)_s = \sum_{i=1}^{N} r_{is} \omega_i \geq d \quad s = 1, \ldots, S \quad (5)$$

$$\sum_{i=1}^{N} \omega_i = 1 \quad (6)$$

$$\omega_i \geq 0, \;\; i = 1, \ldots, N \quad (7)$$

$$l_i \varphi_i \leq \omega_i \leq u_i \varphi_i, \; i = 1, \ldots, N \quad (8)$$

$$\sum_{i=1}^{N} \varphi_i = K \quad (9)$$

$$\varphi_i \in \{0,1\}, i = 1, \ldots, N \quad (10)$$

The scheme of genetic algorithm and method Monte Carlo approach for solving the prospect theory problem with cardinality constrain is given in Fig. 3. Let describe each steps of the approach following the scheme.

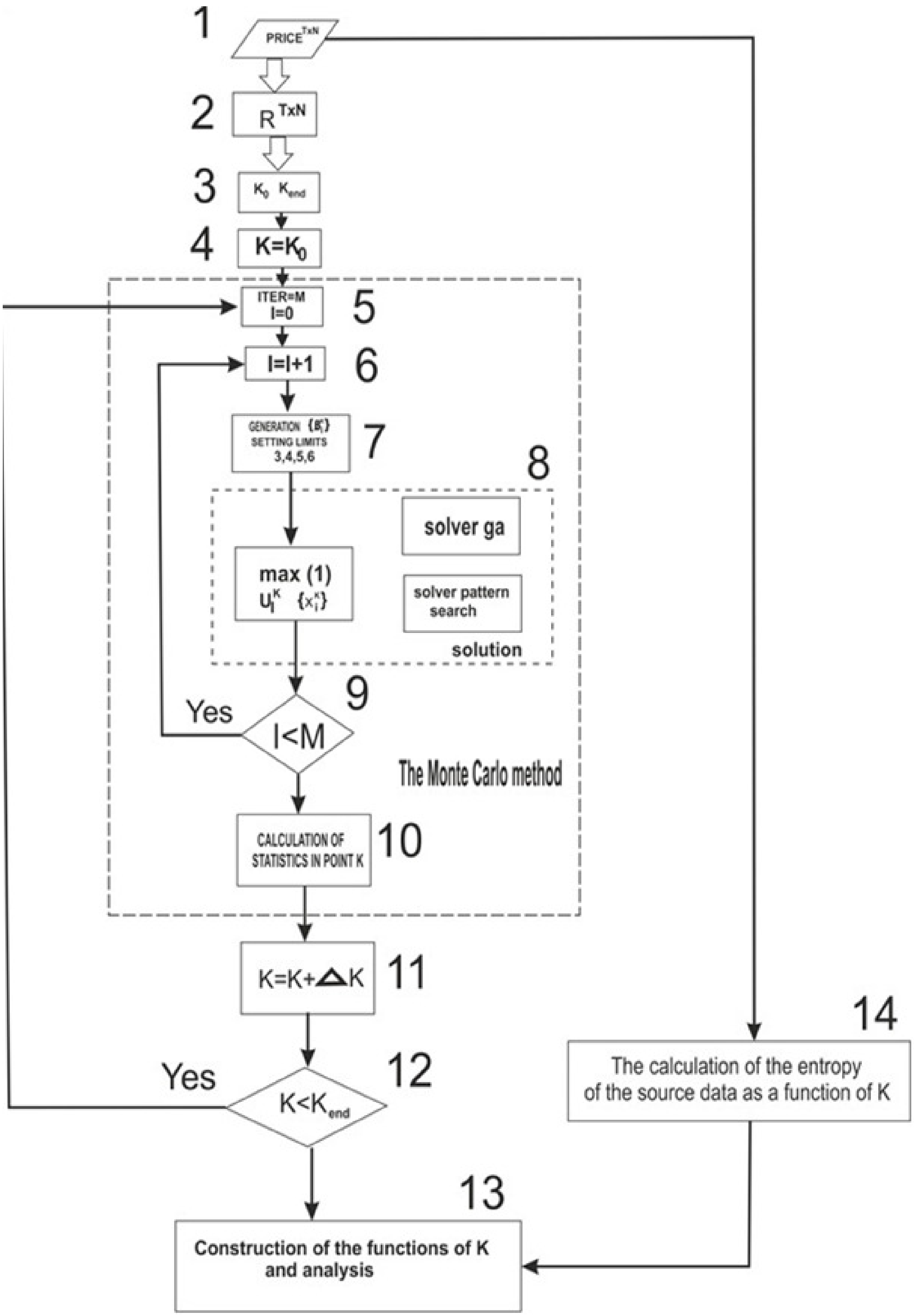

Fig. 3. Scheme of the Prospect Genetic Algorithm
and Method Monte Carlo Approach

In the block 1 we set original information about asset prices in the form of matrix $PRICE^{TXN}$ and then convert it to matrix of returns R^{TXN} (see block 2). In the blocks 3 and 4 is given range of K which is investigated $(K_0 = 0{,}1N, K_{end} = 0{,}75N)$ and initial value of K.The blocks 5-10 implement method Monte Carlo for each value of K from the range of possible values. In block 5 the number of iteration are set and iteration counter is reset to zero.

The number of iteration evaluated according the following assumption. The confident interval of the sample mean is defined as follows:

$$\frac{\sigma_A}{\sqrt{M}} \cdot \left(1 - \sqrt{\frac{M}{M_G}}\right) \cdot t_{0,95}^{M-1} \pm x_m^A \tag{11}$$

where σ_A is the sample standard deviation, M_G is the size of the general population, $t_{0,95}^{M-1}$ is the percentile of of the Student distribution (T-distribution), x_m^A is the sample mean.

In the range of the research it is possible to accept $\left(1 - \sqrt{\frac{M}{M_G}}\right) \approx 1$. Taking into account this assumption and defined (given) value of the confident interval it is possible to evaluate the value of M . The block 6 is a counter of iterations.

In the block 7 for each current iteration randomly generated vector $\beta(i) = 0\ or 1, i = \overline{1, N}$ and set the constraints. In the block 8 the solvers ga (genetic algorithm) and ps (pattern search) search the optimal solution of the problem for the vector $\beta(i) = 0\ or 1, i = \overline{1, N}$ which is generated in the current iteration.

In the block 9 we evaluate the termination condition of iteration. If this condition hold then go to the next iteration and repeat the cycle starting with the block 6. If not, then in the block 10 the algorithm evaluate the values of all investigated parameters in appropriate point K.

In the blocks 11 and 12 the algorithm proceeds to the next point K and evaluates the continuation condition of sensing of the points K (different values of K). If this condition is not held then appropriate functions and graphs are plotted as well as the ratio of the level of uncertainty of the original data and the obtained solutions is analysed (the block 13).

In the block 14 applying the similar scheme the uncertainty of initial data in the form of function of K is analysed. We use entropy as a measure of uncertainty of the data:

$$H = -\sum_{J=1}^{jK} P_j \cdot LOG_2(P_J) \tag{12}$$

where $\sum_{J=1}^{jK} P_j = 1$.

The entropy has maximum value in the case when the uncertainty is complete $P_1 = P_2 = \ldots = P_{JK} = 1/jK$. The entropy of the data is compared with the entropy of the solutions.

From a practical and a theoretical point it is very important to find out how the value of K impact on the statistical properties of the solutions and their relation with the statistical properties of the original data. The algorithm presented above helps to answer these questions.

We tested the performance of our approach for finding the cardinality constrained efficient frontier using publicly available test problems relating to the Hang Seng (Hong Kong) market index, available from OR-Library. The size of the data in $N = 32$ and $S = 290$. The results were obtained for 150 iterations using ga solver and similar results were obtained for 300 iterations using ps solver.

As can be seen in Fig. 4 the level of solution uncertainty is equal to or even greater than the uncertainty of data before cardinality reach the level of 35-40 %, which indicates the low quality of the solutions obtained in this range of cardinality. At higher cardinality model gives a solution with a high level of certainty than in the original data.

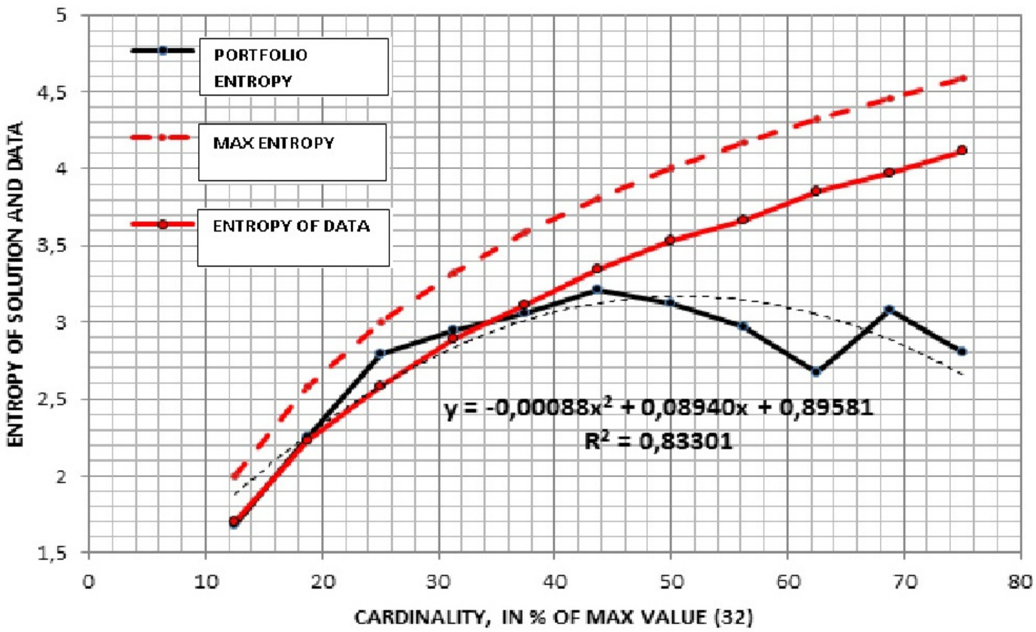

Fig. 4. Analysis of solution uncertainty

In Fig. 5 you can see that with increasing cardinality portfolio volatility first sharply decreases and then increases slightly and stabilized. It means that the stability of the solutions obtained for the cardinality of above 40 %.

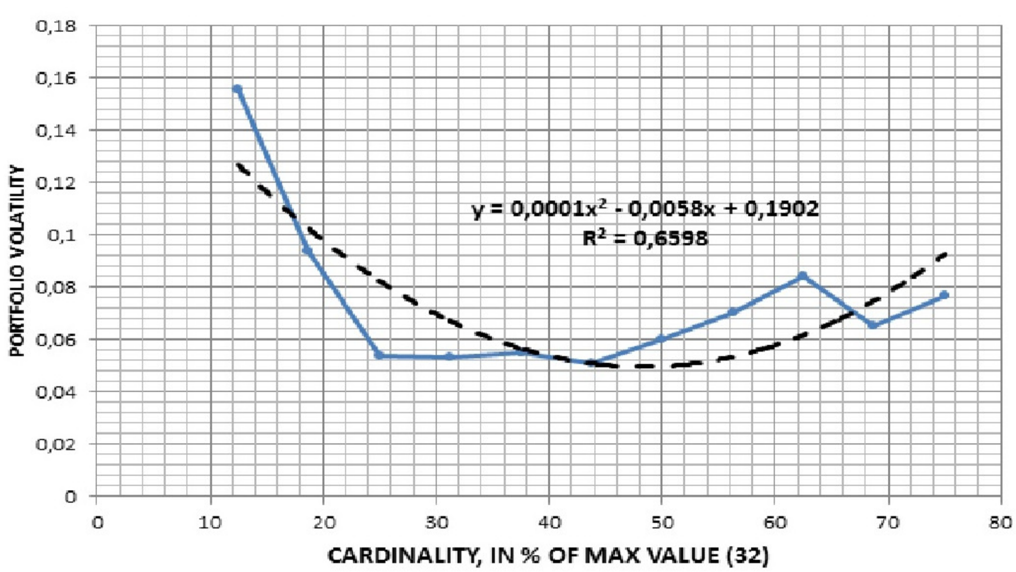

Fig. 5. Analysis Volatility-Cardinality

According to the Fig. 6 skewness increase with the cardinality and asymmetry of the portfolio return distribution shifts to the right (positive part of the graph). Kurtosis is increasing with the cardinality as well and makes the top of the distribution near expected value is sharper. It means that uncertainty is decrease.

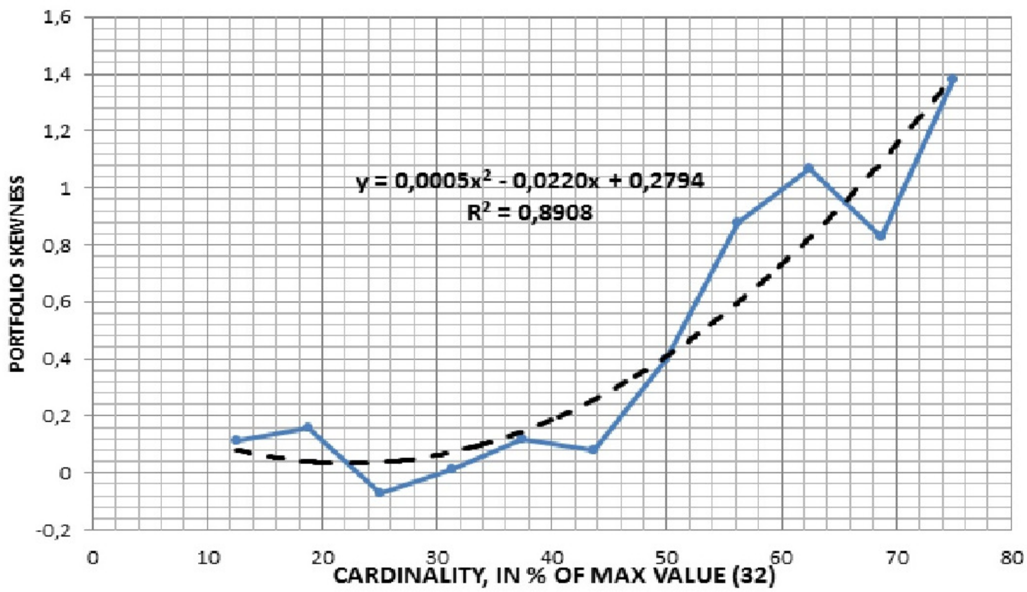

Fig. 6. Analysis Skewness-Cardinality

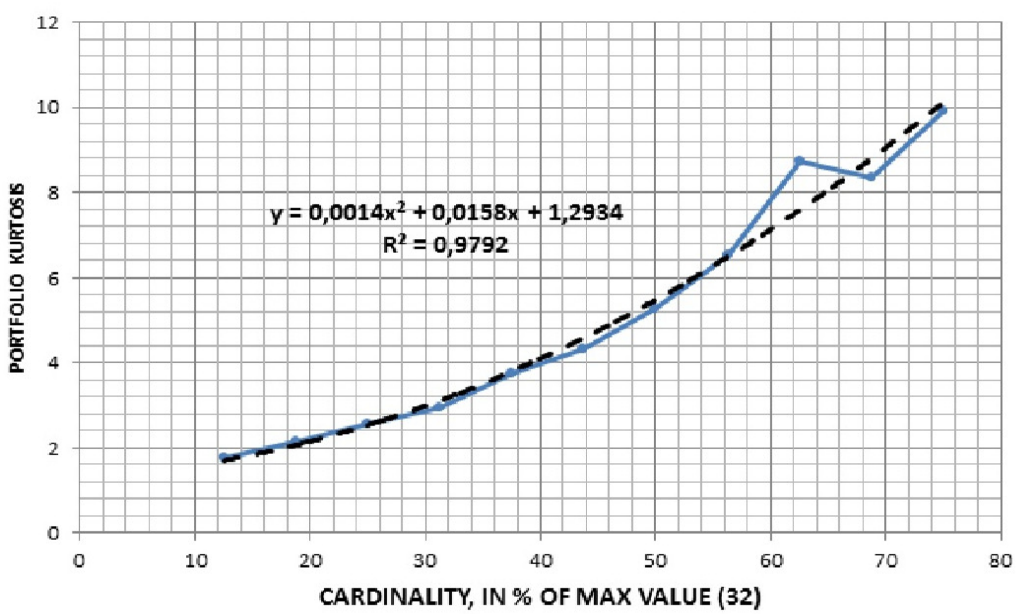

Fig. 7. Analysis Kurtosis-Cardinality

The portfolio selection problem appeared a long time ago when the world financial market is only acquired its modern shape. Thanks to efforts of scientists, general concept of this problem were proposed. Since now investors use their models and measures of risk. At the same time, new needs appear in the market and new models and tools are required. Behavioural portfolio theory which is mainly based on prospect theory is a very challenging theory and a model in the literature in terms of practical benefits of application and solution approaches. However, too many questions were not considered yet. In order to find the right place of behavioral portfolio models it is necessary to develop new computational approaches and to make fundamental researches in this area.

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